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The generation of cross-waves in a long deep channel by parametric resonance

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The evolution equations are obtained which govern the growth of cross-waves generated in a long deep channel by a wavemaker with small amplitude when the waves are modified by finite-amplitude effects. The linearized equations are compared with previous theoretical and experimental work. Some numerical solutions are obtained for illustration.

1. Introduction

Cross-waves are waves that have their crests at right-angles to the wavemaker. Although they have been known for a considerable period of time, their analysis presents great mathematical difficulties owing to the fact that the linearized equations contain no mechanism for getting energy out of the wave. Thus, if energy is transferred directly into the wave by, say, oscillating an asymmetric wavemaker at one of the natural cross-wave frequencies, then the linearized prediction is that the cross-wave will grow indefinitely. However, this does not happen in practice, the difficulty being resolved by the fact that terms which are neglected in the linearized formulation provide a low-level energy drain which, as the wave amplitude grows, eventually becomes comparable to the direct energy feed. This problem has recently received a thorough theoretical and experimental study by Barnard, Mahony & Pritchard (1977).

An even more complex problem is cross-wave generation by a symmetric wavemaker (i.e. one independent of channel width). For this the linearized equation predicts that there is no feed of energy into the cross-wave and hence that it should not be present. Nevertheless, in practice they are observed to exist. Experiments that create cross-waves have their wavemakers oscillated at a frequency one-half that of the natural cross-wave frequency, and this suggests that their generation mechanism is by parametric resonance, i.e. the cross-wave (a small component of which is assumed to be initially present) interacts with the basic motion forced by the wavemaker and by so doing transfers energy into the cross-wave. Provided this influx is greater than any loss of energy which may also be occurring, then the cross-wave grows. Both the energy-gain and the energy-loss mechanisms are relatively low-level affairs, in comparison with the forces represented by the usual linearized water-wave equations, and so we expect to find that the cross-wave has a fixed form, dictated by the linear equations, but to have an amplitude that varies slowly both with time and position along the channel according to the exact details of how the energy is provided.

That the parametric-resonance mechanism is indeed capable of generating cross-waves was first shown by Garrett (1970). He linearized the equations of motion about the amplitude of the disturbance and averaged them in the direction parallel to the crests, eventually obtaining Mathieu's equation, from which he showed that an instability was present with the expected frequency. He also discussed how nonlinear terms should modify his theory. Later Mahony (1972) improved on this analysis by retaining the dependence on the crest direction. At about the same time experiments on cross-waves confirming Mahony's results were performed by Barnard & Pritchard (1972).

Their analysis had the unsatisfactory feature that the linearization about the cross-wave amplitude, rather than the directly forced wave amplitude, obliges one to omit terms in the equation of a larger order of magnitude than those being retained. This was done on the grounds, reasonable at first sight, that they do not have the correct form to affect the resonance. However, although these terms cannot contribute directly, it turns out that they can contribute indirectly, and that linear terms result of the same order of magnitude as those already retained. Also, besides the energy-transfer mechanism delineated by Garrett (work done by pressure forces within the fluid itself), work is also done on the cross-wave directly by the wavemaker itself. In the experiments of Barnard & Pritchard both of these effects were relatively small, which explains their good agreement with previous theory.

The approach in this paper is to retain all the nonlinear terms initially and to develop a uniformly asymptotic solution based on the small amplitude of the wavemaker. This has the additional virtue that the results apply over long periods of time and not just to the initial stages of growth, as with Mahony's analysis. This can be done by first finding the 'natural' solution to the problem (the solution forced by the wavemaker) and then by looking for the cross-wave as the difference between the general solution and this forced solution. By this means, a homogeneous set of equations is obtained for the cross-wave in which the forced solution appears in the coefficients, and such equations can be solved by standard methods. Actually it is not necessary to use the exact forced solution in this procedure – it is sufficient to use merely the first-order approximation to it since the non-homogeneous terms that then result are formally small.

We model the situation as follows. A channel with rigid, vertical walls at $\tilde{y} = 0$ and $\tilde{y} = l$ extends indefinitely in the x-direction and contains water of infinite depth. The positive z-direction is upwards. A wavemaker is located at $x = 0$ and can be represented by

$$\tilde{x} = a\tilde{f}(\tilde{z}) \sin 2\sigma\tilde{t}, \tag{1}$$

where a is the maximum displacement of the wavemaker. The flow is assumed to be incompressible, inviscid and irrotational. Dimensionless variables are defined using a as amplitude scale, g/σ^2 as lengthscale and σ^{-1} as timescale, where g is the acceleration due to gravity. Then the equations to be satisfied are

$$\nabla^2\phi = 0, \tag{2a}$$

$$\frac{\partial\phi}{\partial z} - \frac{\partial\zeta}{\partial t} = \epsilon\frac{\partial\phi}{\partial x_i}\frac{\partial\zeta}{\partial x_i} \quad \text{on} \quad z = \epsilon\zeta, \tag{2b}$$

$$\frac{\partial\phi}{\partial t} + \zeta = -\tfrac{1}{2}\epsilon\frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial x_i} \quad \text{on} \quad z = \epsilon\zeta, \tag{2c}$$

$$\phi \to 0 \quad \text{as} \quad z \to -\infty, \tag{2d}$$

$$\frac{\partial\phi}{\partial x} - \epsilon\frac{\mathrm{d}f}{\mathrm{d}z}\frac{\partial\phi}{\partial z}\sin 2t = 2f(z)\cos 2t \quad \text{on} \quad x = \epsilon f(x)\sin 2t, \tag{2e}$$

$$\frac{\partial\phi}{\partial y} = 0 \quad \text{on} \quad y = 0, l, \tag{2f}$$

and also a radiation condition for large x. In these equations, velocity $\boldsymbol{u} = \nabla\phi$, $\zeta(x, y)$ = surface displacement, $l = b\sigma^2/g$, and $\epsilon = a\sigma^2/g$ (the summation convention applies to subscripted variables). The initial conditions are discussed later.

We make the usual assumption that $\epsilon \ll 1$ (small-amplitude waves). Then there is a first-order solution which one obtains by putting $\epsilon = 0$ in (2) and then solving the linear equations that result. It is

$$\phi = \chi(x, z, t) = A\,\mathrm{e}^{4z}\sin(4x-2t) - \int_0^\infty a(\mu)\left[\cos\mu z + \frac{4}{\mu}\sin\mu z\right]\mathrm{e}^{-\mu x}\,\mathrm{d}\mu\cos 2t, \quad (3a)$$

$$\zeta = \xi(x, t) = -\left.\frac{\partial\chi}{\partial t}\right|_{z=0}, \quad (3b)$$

where

$$A = 4\int_{-\infty}^0 f(s)\,\mathrm{e}^{4s}\,\mathrm{d}s, \quad (3c)$$

$$a(\mu) = \frac{4}{\pi}\frac{\mu}{16+\mu^2}\int_{-\infty}^0 f(s)\left[\cos\mu s + \frac{4}{\mu}\sin\mu s\right]\mathrm{d}s, \quad (3d)$$

and A and $a(\mu)$ are used consistently elsewhere to represent these quantities.

This is expected – a local disturbance near to the wavemaker and further out a wave with crests parallel to the wavemaker. The cross-wave which we expect will be present in addition to this solution, so to distinguish them we write

$$\phi = \psi(x, y, z, t, \epsilon) + \chi, \quad \zeta = \eta(x, y, t, \epsilon) + \xi, \quad (4a, b)$$

and substitute these into the equations of motion (2). Further, we make expansions of the boundary conditions for small ϵ about $z = 0$ and $x = 0$. The equations then become

$$\nabla^2\psi = 0, \quad (5a)$$

$$\frac{\partial\psi}{\partial z} - \frac{\partial\eta}{\partial t} = \epsilon\left\{-(\eta+\xi)\frac{\partial}{\partial z}\left(\frac{\partial\psi}{\partial z} + \frac{\partial\chi}{\partial z}\right) + N(\psi, \eta, \chi, \xi)\right\}$$
$$+\epsilon^2\left\{-\tfrac{1}{2}(\eta+\xi)^2\frac{\partial^2}{\partial z^2}\left(\frac{\partial\psi}{\partial z} + \frac{\partial\chi}{\partial z}\right) + (\eta+\xi)\frac{\partial N}{\partial z}\right\} + O(\epsilon^3) \quad \text{on} \quad z = 0, \quad (5b)$$

where

$$N = \frac{\partial\psi}{\partial x_i}\frac{\partial\eta}{\partial x_i} + \frac{\partial\chi}{\partial x_i}\frac{\partial\eta}{\partial x_i} + \frac{\partial\xi}{\partial x_i}\frac{\partial\psi}{\partial x_i} + \frac{\partial\chi}{\partial x_i}\frac{\partial\xi}{\partial x_i},$$

$$\frac{\partial\psi}{\partial t} + \eta = \epsilon\left\{-(\eta+\xi)\frac{\partial}{\partial z}\left(\frac{\partial\psi}{\partial t} + \frac{\partial\chi}{\partial t}\right) - \tfrac{1}{2}M(\psi, \chi)\right\}$$
$$+\epsilon^2\left\{-\tfrac{1}{2}(\eta+\xi)^2\frac{\partial^2}{\partial z^2}\left(\frac{\partial\psi}{\partial t} + \frac{\partial\chi}{\partial t}\right) - \tfrac{1}{2}(\eta+\xi)\frac{\partial M}{\partial z}\right\} + O(\epsilon^3) \quad \text{on} \quad z = 0, \quad (5c)$$

where

$$M = \frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_i} + 2\frac{\partial\psi}{\partial x_i}\frac{\partial\chi}{\partial x_i} + \frac{\partial\chi}{\partial x_i}\frac{\partial\chi}{\partial x_i},$$

$$\psi \to 0 \quad \text{as} \quad z \to -\infty, \quad (5d)$$

$$\frac{\partial\psi}{\partial x} = \epsilon\left\{-f(z)\sin 2t\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x} + \frac{\partial\chi}{\partial x}\right) + \frac{\mathrm{d}f}{\mathrm{d}z}\left(\frac{\partial\psi}{\partial z} + \frac{\partial\chi}{\partial z}\right)\sin 2t\right\}$$
$$+\epsilon^2\left\{-\tfrac{1}{2}[f(z)\sin 2t]^2\frac{\partial^2}{\partial x^2}\left(\frac{\partial\psi}{\partial x} + \frac{\partial\chi}{\partial x}\right) + f\frac{\mathrm{d}f}{\mathrm{d}z}\sin^2 2t\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial z} + \frac{\partial\chi}{\partial z}\right)\right\} + O(\epsilon^3)$$
$$\text{on} \quad x = 0, \quad (5e)$$

$$\frac{\partial\psi}{\partial y} = 0 \quad \text{on} \quad y = 0, l, \quad (5f)$$

and we wish to solve these for ψ and η. As initial conditions for the problem, we shall assume that the solution is at first close, in some sense, to the linear solution, so that ψ and η are initially small.

2. The possibility of resonance

So far we have been exact. ψ must be at least $O(\epsilon)$ to account for the forcing terms on the right-hand side of (5). However, we also expect to find the equations permit a y-dependent instability, the cross-waves, for which ψ attains a greater order of magnitude. This may be investigated by attempting a regular perturbation expansion of (5) based on some proposed form for the basic cross-wave. If a resonance mechanism is present then sooner or later the higher terms of such an expansion must contain a term that is not periodic. Such a non-uniformity in the regular expansion can have various causes, but when it represents a resonance effect it is interpretable as a means by which energy is being fed slowly into a particular mode through the nonlinear interactions, i.e. by way of the Reynolds stresses. In our particular case it must provide the energy to sustain the cross-wave whose existence we originally proposed. This is a considerable restriction on the resonant modes possible, for then the resonant modes must have the same form as the growing terms, while the structure that one finds for the growing term itself depends on what was initially assumed about the resonant modes. That everything is consistent establishes the basic forms that any resonant modes can take.

If one makes the simplest assumption, that the cross-wave is unforced as a first approximation, then to satisfy (5) it must have the form

$$\psi_0 = \int_0^\infty \int_{-\infty}^\infty B(k,m)\, e^{(k^2+m^2)^{\frac{1}{2}}z} \cos kx\, e^{imy}\, e^{\pm i(k^2+m)^{\frac{1}{2}}t}\, dm\, dk, \tag{6}$$

where B is arbitrary, and the second order of approximation has the equations

$$\nabla^2\psi_1 = 0, \tag{7a}$$

$$\frac{\partial^2\psi_1}{\partial t^2} + \frac{\partial\psi_1}{\partial z} = F(x,y,z,t) \quad \text{on} \quad z = 0, \tag{7b}$$

$$\psi_1 \to 0 \quad \text{as} \quad z \to -\infty, \tag{7c}$$

$$\frac{\partial\psi_1}{\partial x} = G(y,z,t) \quad \text{on} \quad x = 0, \tag{7d}$$

where (7b) is obtained by elimination of η from (5b) and (5c), and F and G are forcing terms which stem from the nonlinear interactions.

These can be solved in principle using transforms. The part of ψ_1 relevant to the instability is

$$\frac{1}{2\pi^3}\int_{-\infty}^\infty \int_{-\infty}^\infty \int_0^\infty \left\{\bar{F} - \int_{-\infty}^0 \frac{\bar{G}(s)\, e^{(k^2+m^2)^{\frac{1}{2}}s}}{2(k^2+m^2)^{\frac{1}{2}}}\, ds\right\} \times \frac{e^{(k^2+m^2)^{\frac{1}{2}}z} \cos kx\, e^{i(my-\omega t)}}{(k^2+m^2)^{\frac{1}{2}} - \omega^2}\, dk\, dm\, d\omega, \tag{8}$$

where $\bar{F}$ and $\bar{G}$ are multiple transforms of F and G.

If we assume that the resonance mechanism occurs by direct interaction of the forced solution with the resonance mode, then this integral must have a singularity for all modal values that are present in (6), i.e. for all values of k and m for which

$B(k, m) \neq 0$. When the appropriate terms in F, the interaction of ψ_0 with the primary wave χ whose form (3.1) is known, is examined we find that any wave packet $\cos Kx\, e^{iMy}\, e^{i\Omega t}$ of the resonant mode will have its y-dependence unchanged by interaction, while the frequency of the wave packet is shifted by two units. Thus if (8) is to produce a term of the same form as the original wave packet we must certainly have $\Omega = \pm 1$. Examining (8) further, it is easy to verify that the interaction of such modes with the wave component of χ cannot lead to any resonance, while interaction with the remaining part of χ leads to a continuous k-spectrum for $\bar{F}$ in (8). But then the integral can never be sufficiently singular for our purposes unless $\Omega^2 = \omega^2 = |M|$, whereupon there is a singularity of $O(k^{-2})$ and not $O(k^{-1})$ at $k = 0$. That the singularity in (8) is at $k = 0$ tells us that $K = 0$ so the first-order approximation to the cross-wave can only be of the form

$$\psi_0 = \mathrm{Re}\{(\alpha_0\, e^{iy} + \beta_0\, e^{-iy})\, e^{z-it}\}. \tag{9}$$

The result $K = 0$ could be expected on physical grounds since this is the only wavenumber for which the group velocity in the x-direction is zero (to first order). This is necessary because the energy feed into the wave is small, occurring via the interaction of the small nonlinear terms. Thus a non-zero group velocity, which would allow the energy to escape at an order of magnitude faster than it is supplied, cannot be possible.

This section has been dealt with briefly: similar results were obtained by Mahony (1972) by use of integral equations.

3. The asymptotic expansion

So far we have only established that a resonant mode of the form (9) has the possibility of existence, and it remains to be shown that it actually does exist. We shall establish this using a version of the method of multiple scales.

The functions α_0 and β_0 in (9), only constant as a first approximation, vary slowly with x, y, z and t. The first task is to determine the correctly scaled variables, and also the order of magnitude of the resonant mode ψ_0. This is done in the usual manner of selecting scalings which, in higher-order equations, produce balanced terms for the resonance component. Strictly, then, we must determine how the higher-order equations behave before we can begin. For presentation purposes it is easier to just quote the results that we need, and the reader can relate them to later sections as he comes to them.

The resonance equation is essentially the Fourier (x) transform of the ($\cos y$)-component of the $O(\epsilon^2)$ terms of (5*b*) and (5*c*). This is not obvious since the forcing term found above from the direct interaction of ψ_0 and χ is $O(\epsilon)$, and one would expect the solution of the problem to occur at this order. However, it is the Fourier transform that matters, and the Fourier transform of such products is smaller by an order of magnitude than the Fourier transform of terms which depend solely on the scaled variable. If the scaled x-variable is X, and the scaled t-variable is τ, then the terms that one obtains for comparison of orders of magnitude are

$$\int_0^\infty \left\{\frac{\tau}{t}\frac{\partial^2\psi_0}{\partial t\,\partial\tau} : \left(\frac{X}{x}\right)^2 \frac{\partial^2\psi_0}{\partial X^2} : \epsilon\psi_0\chi : \epsilon^2\psi_0^3\right\} dx, \tag{10}$$

where the first of these terms comes directly from the linear terms of (5*b*, *c*), the second is the contribution to $\partial\psi/\partial z$ from (5*a*), the third is the forcing term and the fourth is the second interaction of the resonance mode. The third term depends on x and

X, while the others depend on X only. For equal importance of all terms

$$X = \epsilon x, \quad \tau = \epsilon^2 t \tag{11}$$

and $\psi_0 = O(1)$, from which we may propose the expansion

$$\begin{aligned} \psi &\sim \psi_0 + \epsilon\psi_1 + \epsilon^2\psi_2 + \dots, \\ \eta &\sim \eta_0 + \epsilon\eta_1 + \epsilon^2\eta_2 + \dots . \end{aligned} \tag{12}$$

There is no need for a stretched z-scale, since the expansion is regular in the z-direction. A more complicated situation results from considering the y-direction. As long as the region $[0, l]$ is bounded, the expansion will be regular in y, and so it is possible to proceed without introduction of a scaled variable. However, to satisfy the conditions (5f) on the sidewalls, $\alpha_0 = \beta_0$, so that

$$\psi_0 = \cos y \, \mathrm{e}^z \, [C(X, \tau) \cos t + D(X, \tau) \sin t], \tag{13}$$

while also the wall separation must be a multiple of the wavelength, $l = n\pi$ ($n = 1, 2, 3, \dots$). This will not be exactly true in a general situation, but solutions are still possible as long as l/π is sufficiently close to an integer. 'Close' is defined by the requirement that the side boundaries have no effect greater than those represented by the terms in (10), and this implies that

$$(1 + \lambda\epsilon^2)\, l = n\pi, \tag{14}$$

where λ can be constant $O(1)$. Thus λ acts as an arbitrary detuning parameter. This relationship defines a unique cross-wave providing the channel width l is $o(\epsilon^{-2})$, but, when l is even bigger than this, more than one value for n, and hence λ, is possible. Indeed, if l were to become infinite, a continuous spectrum could be obtained for λ, and one sidewall condition would have to be replaced by a radiation condition. It is clear what is happening here: it is the side boundary conditions that determine the cross-wavelength, but for resonance to exist this wavelength must be very close to 2π. This determines the wavelength uniquely unless the channel breadth is exceptionally large, when more than one such wavelength is possible.

We shall ignore the interesting possibilities of large l, and confine our attention to a single cross-wave mode by taking $l = O(1)$. However, since the channel width appears to be a natural length for the problem we shall introduce a variable based on it by

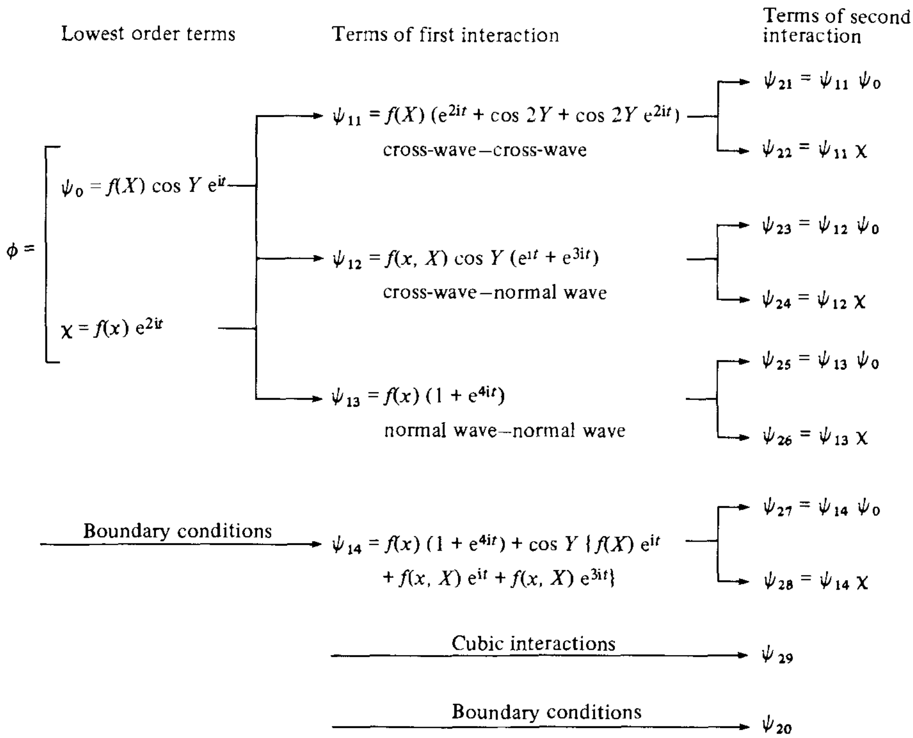
$$Y = \frac{n\pi}{l} y = (1 + \lambda\epsilon^2)\, y, \tag{15}$$

and then eliminate y from the equations in favour of Y. (The treatment is distinctly different to that of x and t, where the scaled variables X and τ are introduced *in addition* to x and t.) The changes produced are minor: (5a) is affected at second order, while the side boundary conditions become easy to apply.

The new variables (11) are added to the equations (5) in the normal way by writing $\partial/\partial x + \epsilon\, \partial/\partial X$ for $\partial/\partial x$, and $\partial/\partial t + \epsilon^2\, \partial/\partial \tau$ for $\partial/\partial t$. The expansions (12) are also substituted into the equations, and terms with equal orders of magnitude are equated to one another. The equations as far as $O(\epsilon^2)$ are needed and are listed later as they are solved.

4. Development of the equations

The sets of linear equations that result can in principle be solved, but in practice present immense labour. Fortunately, only a small fraction of the full solution need be computed if obtaining a uniformly valid first expansion for the cross-wave ψ_0 is our sole objective. Consider the following diagram, which represents schematically how the asymptotic series develops by nonlinear interactions.



In the diagram, as in the discussion, ψ represents both ψ and η, and χ represents both χ and ξ. One must keep in mind that the final equation for ψ_0 is the Fourier transform of the cos Y component of the first time harmonic. Such terms, with the correct order of magnitude, can only be obtained in a limited number of ways.

The zeroth-order term is ψ_0. The first-order term is ψ_1, and the components of this are generated in one of four ways. The parts of it are (i) ψ_{11}, formed by ψ_0 interacting with itself, (ii) ψ_{12}, formed by ψ_0 interacting with χ, (iii) ψ_{13}, formed by χ interacting with itself, and (iv) ψ_{14}, which originates with the boundary condition at $x = 0$. The side boundary conditions are automatically satisfied at this order and do not produce any terms. The different forms that these components take are illustrated in the diagram. Neither ψ_{11} nor ψ_{13} is able to contribute immediately to the resonance equation whereas both ψ_{12} and ψ_{14} contain a singularity which directly affects it. These singularities, which have the same form, could be treated together, but are dealt with in different sections to help comparison with earlier work. The singularity in ψ_{12} was noticed by earlier workers, but the singularity in ψ_{14} was overlooked. They are both of the type discussed in §2. The procedure adopted is conventional whereby prevention of the singularity determines the value of $\partial\psi_0/\partial X$ at $X = 0$, which thus furnishes the boundary condition of the final equation for ψ_0.

In the next stage of the analysis, at second order in ϵ, the only relevant terms are functions of X only, completely independent of the short scale x. They must also have factors of cos Y and e^{it}. Such terms cause a different type of singularity to that found in the first-order terms, and its removal produces the resonance equation. Once again, however, terms independent of x and with the correct (Y, t)-dependence can only be obtained from the second-order interactions in a limited number of ways. We now discuss these in detail.

Of the ten components of ψ_2 in the diagram, eight are generated by interaction of ψ_1 with either ψ_0 or χ. If the interaction is with ψ_0, which is independent of x, we need only have retained terms in ψ_1 which themselves are independent of x. If the interaction is with χ, we need only consider the interaction with its wave component, and since this has a factor of e^{4ix} it follows that we need only retain the terms in ψ_1 that have a factor to cancel this. These two types are the only terms that need to be explicitly evaluated when calculating ψ_1, and this substantially reduces the labour involved. It turns out that ψ_{11} has terms independent of x, and so ψ_{21} contains useful terms, while ψ_{12} has a factor of e^{-4ix}, and thus ψ_{24} must be determined. None of the other six components ψ_{22}, ψ_{23}, ψ_{25}, ψ_{26}, ψ_{27}, ψ_{28}, however, need be calculated either because the ψ_1 component has no useful x-behaviour, or because the time dependence is not right.

Besides the above, there are two other possible sources of ψ_2 components. Terms in ψ_{29} are forced by triple interaction of ψ_0 and χ, but here again only a limited number of terms need be retained, and the part independent of x is easy to calculate. The final term ψ_{20} is the component forced by the side boundary conditions. The boundary condition at the wavemaker causes a singularity of a different kind and can be neglected.

Thus it is rare that any term has to be found in full. Because of this the equations are not presented *en bloc*. Instead, equations for each component are presented separately and then solved to obtain the fraction needed. Where the full solution is not calculated the expression that contains the relevant part of that particular component is denoted by an asterisk. Similarly, the wave component of χ (or ξ), which is the only part capable of producing an x-independent term, is denoted by χ^* (or ξ^*).

5. The lowest-order terms

The equations are
$$\nabla^2\psi_0 = 0, \tag{16a}$$
$$\frac{\partial\psi_0}{\partial z} - \frac{\partial\eta_0}{\partial t} = 0 \quad \text{on} \quad z = 0, \tag{16b}$$
$$\frac{\partial\psi_0}{\partial t} + \eta_0 = 0 \quad \text{on} \quad z = 0, \tag{16c}$$
with other boundary conditions homogeneous. The solution is
$$\psi_0 = \cos Y(C(X,\tau)\cos t + D(X,\tau)\sin t)\,e^z, \tag{17a}$$
$$\eta_0 = \cos Y(C(X,\tau)\sin t - D(X,\tau)\cos t). \tag{17b}$$

6. The determination of ψ_{11} ($\psi_0\,\psi_0$ terms)

The equations are
$$\nabla^2\psi_{11} = 0, \tag{18a}$$
$$\frac{\partial\psi_{11}}{\partial z} - \frac{\partial\eta_{11}}{\partial t} = -\eta_0\frac{\partial^2\psi_0}{\partial z^2} + \frac{\partial\psi_0}{\partial Y}\frac{\partial\psi_0}{\partial Y} \quad \text{on} \quad z = 0, \tag{18b}$$
$$\frac{\partial\psi_{11}}{\partial t} + \eta_{11} = -\eta_0\frac{\partial^2\psi_0}{\partial z\,\partial t} - \frac{1}{2}\frac{\partial\psi_0}{\partial Y}\frac{\partial\psi_0}{\partial Y} - \frac{1}{2}\frac{\partial\psi_0}{\partial z}\frac{\partial\psi_0}{\partial z} \quad \text{on} \quad z = 0, \tag{18c}$$

with other boundary conditions homogeneous. It is straightforward to solve these equations after substitution for ψ_0 and η_0 from (17). The solution is

$$\psi_{11} = \tfrac{1}{4}\{2CD\cos 2t + (D^2 - C^2)\sin 2t\}, \tag{19a}$$

$$\eta_{11} = \tfrac{1}{4}\cos 2Y\{(D^2-C^2)\cos 2t - 2CD\sin 2t + (D^2+C^2)\}. \tag{19b}$$

7. The determination of ψ_{12} ($\psi_0\chi$ terms)

The equations are

$$\nabla^2\psi_{12} = 0, \tag{20a}$$

$$\frac{\partial\psi_{12}}{\partial z} - \frac{\partial\eta_{12}}{\partial t} = -\eta_0\frac{\partial^2\chi}{\partial z^2} - \xi\frac{\partial^2\psi_0}{\partial z^2} \quad \text{on} \quad z = 0, \tag{20b}$$

$$\frac{\partial\psi_{12}}{\partial t} + \eta_{12} = -\eta_0\frac{\partial^2\chi}{\partial z\,\partial t} - \xi\frac{\partial^2\psi_0}{\partial z\,\partial t} - \frac{\partial\psi_0}{\partial z}\frac{\partial\chi}{\partial z} \quad \text{on} \quad z = 0, \tag{20c}$$

$$\frac{\partial\psi_{12}}{\partial x} = -\frac{\partial\psi_0^{(1)}}{\partial X} \quad \text{on} \quad x = X = 0.$$

where we are partitioning ψ_0 into two parts:

$$\psi_0 = \psi_0^{(1)} + \psi_0^{(2)} = \cos Y e^z\,[(C^{(1)} + C^{(2)})\cos t + (D^{(1)} + D^{(2)})\sin t], \tag{21}$$

since exactly the same form of singularity will appear when calculating ψ_{14} in §9. The distinction between these singularities is, of course, entirely artificial, being a consequence of the way we have grouped second-order terms together in blocks related to the way those terms originate, and only the totality of all the terms together has any real significance. Thus the form of ψ_0 must be such as to eliminate the total singularity in ψ_{12} and ψ_{14}. We arrange for this to happen by defining $\psi_0^{(1)}$ to be the part of ψ_0 that eliminates the singularity in ψ_{12}, and $\psi_0^{(2)}$ to be the part that eliminates the singularity in ψ_{14}. This allows us to continue with our conceptual differentiation of the two types of terms. Although the conditions that we obtain on the separate functions $\psi_0^{(1)}$ and $\psi_0^{(2)}$ then have no individual significance their sum will produce a real condition which must be satisfied by the function ψ_0.

The other boundary conditions are homogeneous.

We begin by eliminating η_{12} from (20*b*, *c*), and continue by substituting for χ and ξ from (3). Since $\eta_0 = -[\partial\psi_0/\partial t]_{z=0}$, one thus obtains

$$\begin{aligned}\frac{\partial^2\psi_{12}}{\partial t^2} + \frac{\partial\psi_{12}}{\partial z} = &-8A\frac{\partial\psi_0}{\partial t}\sin(4x-2t) + 16A\psi_0\cos(4x-2t)\\ &+ \frac{\partial\psi_0}{\partial t}\cos 2t\int_0^\infty(\mu^2+24)\,a(\mu)\,e^{-\mu x}\,d\mu\\ &- 16\psi_0\sin 2t\int_0^\infty a(\mu)\,e^{-\mu x}\,d\mu \quad \text{on} \quad z = 0.\end{aligned} \tag{22a}$$

Since the non-homogeneous terms all have a factor of cos Y, we may assume that the same is true of the solution. Thus if we now take the half-range Fourier cosine transform (which we denote by a bar over the variable) of (20*a*) then it can be solved to show that

$$\bar{\psi}_{12} = h(k,t;X)\cos Y e^{(1+k^2)^{\frac{1}{2}}z} + \frac{1}{k^2}\left[\frac{\partial\psi_0^{(1)}}{\partial X}\right]_{X=0}, \tag{22b}$$

where $h(k,t;X)$ is arbitrary.

This enables us to take the transforms of (20d), which, if we also substitute for $a(\mu)$, becomes

$$\cos Y\left[\frac{\partial^2 h}{\partial t^2}+(1+k^2)^{\frac{1}{2}}h\right]=4A\frac{\partial\psi_0}{\partial t}\left[\pi\delta(4-k)\sin 2t-\frac{8\cos 2t}{16-k^2}\right]$$
$$+8A\psi_0\left[\pi\delta(4-k)\cos 2t+8\sin\frac{2t}{16-k^2}\right]$$
$$+\frac{\partial\psi_0}{\partial t}\cos 2t\left\{2f(0)+2\int_{-\infty}^{0}f(s)\left[\frac{64}{16-k^2}e^{4s}+\frac{k^3+4k^2-24k-96}{16-k^2}e^{ks}\right]ds\right\}$$
$$-32\psi_0\sin 2t\int_{-\infty}^{0}f(s)\left[\frac{8e^{4s}}{16-k^2}+\frac{e^{ks}}{k-4}\right]ds \quad \text{on} \quad z=0.$$

Finally we substitute explicitly for ψ_0 from (17a), and so obtain first and third time-harmonic factors on the right-hand side, factors which we assume to be correspondingly present in the solution forced by these terms. The third harmonic causes no problem, but the first harmonic gives rise to a singularity in the Fourier transform (22a) at $k=0$. This is in fact part of the same singularity found earlier in §2, and it is important in forcing the cross-wave.

To prevent singular behaviour at this order we have to select the value of $\partial\psi_0^{(1)}/\partial X$ at $X=0$ so that the singularity vanishes. For this to happen, we must choose

$$\frac{\partial\psi_0^{(1)}}{\partial X}=H(C\sin t+D\cos t)\cos Y e^z \quad \text{at} \quad X=0, \tag{23a}$$

where
$$H=4\int_{-\infty}^{0}f(s)\,ds-2f(0). \tag{23b}$$

Once this is done, the solution is regular, and in principle may be calculated. Fortunately only a small part of the solution need be evaluated, since most of the terms cannot interact at next order to produce a resonance term. The first and third harmonic factors that we have at present would have to react with χ in order to obtain the correct time dependence. Further, only reaction with χ^* can give terms independent of x, and this is only possible if there is an e^{4ix} factor to eliminate the x-dependence of χ^*. Such terms do exist: in fact, as might be expected, they are the terms forced in the present-order solution by χ^* reacting with ψ_0. They are

$$\psi_{12}^*=A\cos Y e^{\sqrt{17}z}\{\tfrac{1}{4}(1+\sqrt{17})[C\cos(4x-t)+D\sin(4x-t)]$$
$$-\tfrac{1}{16}(27+3\sqrt{17})[C\cos(4x-3t)-D\sin(4x-3t)]\}, \tag{24a}$$
$$\eta_{12}^*=A\cos Y\{\tfrac{1}{4}(11-\sqrt{17})[C\sin(4x-t)-D\cos(4x-t)]$$
$$+\tfrac{1}{16}(-31+9\sqrt{17})[C\sin(4x-3t)+D\cos(4x-3t)]. \tag{24b}$$

8. The determination of ψ_{13} (χ^2 terms)

We need only calculate the steady component, since the 4th-harmonic cannot contribute at next order. Thus the equations reduce to

$$\nabla^2\psi_{13}^*=0, \tag{25a}$$
$$\frac{\partial\psi_{13}^*}{\partial z}=\text{steady part of}\left[\frac{\partial\chi}{\partial x}\frac{\partial\xi}{\partial x}-\xi\frac{\partial^2\chi}{\partial z^2}\right] \quad \text{on} \quad z=0, \tag{25b}$$

$$\eta_{13}^* = \text{steady part of } -\frac{1}{2}\left[\left(\frac{\partial\chi}{\partial x}\right)^2 + \left(\frac{\partial\chi}{\partial z}\right)^2 - \xi\frac{\partial^2\chi}{\partial z\,\partial t}\right] \quad \text{on} \quad z = 0, \tag{25c}$$

and other boundary conditions are homogeneous.

We now substitute for χ and ξ from (3), and the equations are solved as usual by a Fourier cosine transform. ψ_{13}^* turns out to have a logarithmic singularity for large x. However, to contribute at next order it is $\nabla\psi_{13}^*$ which must remain finite for large x. It obviously decays for $x = O(1)$, and so we can write

$$\psi_{13}^* = 0, \tag{26a}$$

while η_{13}^* is easily calculated from (25c) to be also zero for large x. So

$$\eta_{13}^* = 0. \tag{26b}$$

9. The determination of ψ_{14} (the boundary terms)

The equations are

$$\nabla^2\psi_{14} = -2\frac{\partial^2\psi_0}{\partial x\,\partial X} = 0, \tag{27a}$$

$$\frac{\partial\psi_{14}}{\partial z} - \frac{\partial\eta_{14}}{\partial t} = 0 \quad \text{on} \quad z = 0, \tag{27b}$$

$$\frac{\partial\psi_{14}}{\partial t} + \eta_{14} = 0 \quad \text{on} \quad z = 0, \tag{27c}$$

$$\frac{\partial\psi_{14}}{\partial x} = -f(z)\sin 2t\left[\frac{\partial^2\psi_0}{\partial x^2} + \frac{\partial^2\chi}{\partial x^2}\right] + \frac{\mathrm{d}f}{\mathrm{d}z}\left[\frac{\partial\psi_0}{\partial z} + \frac{\partial\chi}{\partial z}\right]\sin 2t - \frac{\partial\psi_0^{(2)}}{\partial X} \quad \text{on} \quad x = X = 0, \tag{27d}$$

$$\frac{\partial\psi_{14}}{\partial Y} = 0 \quad \text{on} \quad Y = 0, n\pi, \tag{27e}$$

$$\psi_{14} \to 0 \quad \text{as} \quad z \to -\infty. \tag{27f}$$

We pursue the usual course of (i) explicitly substituting for the non-homogeneous terms, (ii) assuming the (t, Y)-dependence of the solution is the same as that of the forcing terms, (iii) applying the Fourier cosine transform to (27a), (iv) determining any arbitrary functions from the surface boundary conditions (27b, c).

It is unnecessary to calculate the fourth time-harmonic component of the solution since it cannot interact at next order to yield a resonance term. The third harmonic could contribute, however, and must be calculated. For large values of x it can be shown to satisfy

$$\text{third harmonic of } \psi_{14} \sim \text{constant}\, \mathrm{e}^{\mathrm{i}(\sqrt{80}\,x - 3t)}\cos Y \mathrm{e}^{9z}, \tag{28}$$

and so this also cannot interact to form any resonance terms, since they must be independent of the coordinate x, and hence it can be neglected. Similarly we may neglect the steady component of ψ_{14} since it decays to zero over a distance $x = O(1)$. There is no second harmonic component to be looked at. The remaining discussion concerns the first harmonic component only: the boundary condition (27d) becomes

$$\left[\frac{\partial\psi_{14}^*}{\partial x}\right]_{x=0} = \cos T\mathrm{e}^z\left\{\frac{1}{2}\frac{\mathrm{d}f}{\mathrm{d}z}(C_0\sin t + D_0\cos t) - \left(\frac{\partial C_0^{(2)}}{\partial X}\cos t + \frac{\partial D_0^{(2)}}{\partial X}\sin t\right)\right\}, \tag{29}$$

where a zero subscript on C or D denotes its value at $X = 0$. Equation (29) can be used to take the Fourier transform of (27a), and it is

$$\frac{\partial^2 \bar{\psi}^*_{14}}{\partial z^2} - (k^2+1)\,\bar{\psi}^*_{14} = \cos Y\, \mathrm{e}^z \left\{ \frac{1}{2}\frac{\mathrm{d}f}{\mathrm{d}z}(C_0 \sin t + D_0 \cos t) - \left(\frac{\partial C_0^{(2)}}{\partial X}\cos t + \frac{\partial D_0^{(2)}}{\partial X}\sin t \right) \right\}. \tag{30}$$

The general solution of this which satisfies the condition (27f) is

$$\begin{aligned}\psi^*_{14} = {} & \frac{\cos Y}{(k^2+1)^{\frac{1}{2}}} \left\{ \int_0^z \frac{1}{2}\frac{\mathrm{d}f}{\mathrm{d}s}\mathrm{e}^s \sinh\left[(k^2+1)^{\frac{1}{2}}(z-s)\right] \mathrm{d}s \right. \\ & \left. + \mathrm{e}^{-(k^2+1)^{\frac{1}{2}} z} \int_0^{-\infty} \frac{1}{4}\frac{\mathrm{d}f}{\mathrm{d}s}\mathrm{e}^s\, \mathrm{e}^{(k^2+1)^{\frac{1}{2}} s}\, \mathrm{d}s \right\} (C_0 \sin t + D_0 \cos t) \\ & + \frac{\cos Y\, \mathrm{e}^z}{k^2}\left(\frac{\partial C_0^{(2)}}{\partial X}\cos t + \frac{\partial D_0^{(2)}}{\partial X}\sin t \right) + B(k, Y, t)\, \mathrm{e}^{(k^2+1)^{\frac{1}{2}} z},\end{aligned} \tag{31}$$

and the function B_0 is determined from the surface boundary conditions (27b,c). Application of these yields

$$B = \frac{[1+(k^2+1)^{\frac{1}{2}}]^2 \cos Y}{4k^2(k^2+1)^{\frac{1}{2}}} \int_0^{-\infty} \frac{\mathrm{d}f}{\mathrm{d}s}\mathrm{e}^{[1+(k+1)^{\frac{1}{2}}]s}\, \mathrm{d}s\, (C_0 \sin t + D_0 \cos t). \tag{32}$$

As in §7, the transform (31) must not be singular at $k = 0$, and to avoid this we are now forced to choose

$$\frac{\partial \psi_0^{(2)}}{\partial X} = K \cos Y\, \mathrm{e}^z (C \sin t + D \cos t) \quad \text{at} \quad X = 0, \tag{33a}$$

where
$$K = \int_{-\infty}^{0} \frac{\mathrm{d}f}{\mathrm{d}s}\mathrm{e}^{2s}\, \mathrm{d}s. \tag{33b}$$

If we combine this result with (23), we have

$$\frac{\partial \psi_0}{\partial X} = L \cos Y\, \mathrm{e}^z (C \sin t + D \cos t) \quad \text{at} \quad X = 0, \tag{34a}$$

where
$$L = H + K = \int_{-\infty}^{0} \left(4f(s) + \frac{\mathrm{d}f}{\mathrm{d}s}\mathrm{e}^{2s}\right) \mathrm{d}s - 2f(0). \tag{34b}$$

Equivalently we may express this as

$$\frac{\partial C_0}{\partial X} = LD_0, \quad \frac{\partial D_0}{\partial X} = LC_0, \tag{34c}$$

and so we are furnished with the boundary conditions that we shall need later after the resonance equation has been obtained.

It is now necessary to continue and actually invert $\bar{\psi}^*_{14}$, since interactions with ψ^*_{14} at next order do not produce any terms of resonance form. The only possibility which could achieve the correct time dependence is an interaction with χ^*, but χ^* contains an e^{4ix} factor which ψ_{14} is unable to cancel.

We must also consider whether there are any free modes in the solution for ψ_1. The only possibility, of course, is a mode of resonant form, since other modes would radiate their energy away too quickly. However, a resonant mode in ψ_1 can be neglected, since it cannot interact at next order to produce any useful term.

10. The resonance equation

Now that all the relevant first-order terms in the asymptotic expansion have been determined, we can continue to second order. Only those terms which have the resonance form need to be retained in the equations, which are thus

$$\nabla^2\psi_2^* = -\frac{\partial^2\psi_0}{\partial X^2} - 2\frac{\partial^2\psi_1}{\partial x\partial X} + 2\lambda\frac{\partial^2\psi_0}{\partial Y^2}, \tag{35a}$$

$$\begin{aligned}\frac{\partial\psi_2^*}{\partial z} - \frac{\partial\eta_2^*}{\partial t} - \frac{\partial\eta_0}{\partial\tau} = {} & -\eta_{11}\frac{\partial^2\psi_0}{\partial z^2} + \frac{\partial\psi_{11}}{\partial Y}\frac{\partial\eta_0}{\partial Y} + \frac{\partial\eta_{11}}{\partial Y}\frac{\partial\psi_0}{\partial Y} - \tfrac{1}{2}\eta_0^2\frac{\partial^3\psi_0}{\partial z^3} + \eta_0\frac{\partial^2\psi_0}{\partial z\,\partial Y}\frac{\partial\eta_0}{\partial Y}\\ & -\eta_{12}\frac{\partial^2\chi}{\partial z^2} - \xi\frac{\partial^2\psi_{12}}{\partial z^2} + \frac{\partial\chi}{\partial x}\frac{\partial\eta_{12}}{\partial x} + \frac{\partial\xi}{\partial x}\frac{\partial\psi_{12}}{\partial x}\\ & -\tfrac{1}{2}\xi^2\frac{\partial^3\psi_0}{\partial z^3} - \eta_0\xi\frac{\partial^3\chi}{\partial z^3} + \eta_0\frac{\partial^2\chi}{\partial z\,\partial x}\frac{\partial\xi}{\partial x} \quad \text{on} \quad z = 0,\end{aligned} \tag{35b}$$

$$\begin{aligned}\frac{\partial\psi_2^*}{\partial t} + \eta_2^* - \frac{\partial\psi_0}{\partial\tau} = {} & -\eta_{11}\frac{\partial^2\psi_0}{\partial z\,\partial t} - \frac{\partial\psi_{11}}{\partial Y}\frac{\partial\psi_0}{\partial Y} - \frac{\partial\psi_{11}}{\partial z}\frac{\partial\psi_0}{\partial z} - \tfrac{1}{2}\eta_0^2\frac{\partial^3\psi_0}{\partial z^2\,\partial t} - \eta_0\frac{\partial^2\psi_0}{\partial z\,\partial y}\frac{\partial\psi_0}{\partial Y}\\ & -\eta_0\frac{\partial^2\psi_0}{\partial z^2}\frac{\partial\psi_0}{\partial z} - \eta_{12}\frac{\partial^2\chi}{\partial z\,\partial t} - \xi\frac{\partial^2\psi_{12}}{\partial z\,\partial t} - \frac{\partial\psi_{12}}{\partial x}\frac{\partial\chi}{\partial x} - \frac{\partial\psi_{12}}{\partial z}\frac{\partial\chi}{\partial z}\\ & -\tfrac{1}{2}\xi^2\frac{\partial^3\psi_0}{\partial z^2\,\partial t} - \eta_0\xi\frac{\partial^3\chi}{\partial z^2\,\partial t} - \eta_0\frac{\partial^2\chi}{\partial z\,\partial x}\frac{\partial\chi}{\partial x}\\ & -\eta_0\frac{\partial^2\chi}{\partial z^2}\frac{\partial\chi}{\partial z} - \xi\frac{\partial^2\psi_0}{\partial z^2}\frac{\partial\chi}{\partial z} - \xi\frac{\partial\psi_0}{\partial z}\frac{\partial^2\chi}{\partial z^2} \quad \text{on} \quad z = 0,\end{aligned} \tag{35c}$$

$$\psi_2 \to 0 \quad \text{as} \quad z \to -\infty, \tag{35d}$$

$$\begin{aligned}\frac{\partial\psi_2}{\partial x} = \sin 2t\left[-f(z)\frac{\partial^2\psi_1}{\partial x^2} + \frac{\mathrm{d}f}{\mathrm{d}z}\frac{\partial\psi_1}{\partial z}\right] + \sin^2 2t\left[-\tfrac{1}{2}f^2(z)\frac{\partial^2}{\partial x^2}\left(\frac{\partial\psi_0}{\partial x} + \frac{\partial\chi}{\partial x}\right)\right.\\ \left. + f\frac{\mathrm{d}f}{\mathrm{d}z}\frac{\partial}{\partial x}\left(\frac{\partial\psi_0}{\partial z} + \frac{\partial\chi}{\partial z}\right)\right] - \frac{\partial\psi_1}{\partial X} \quad \text{on} \quad x = X = 0,\end{aligned} \tag{35e}$$

$$\frac{\partial\psi_2}{\partial Y} = 0 \quad \text{on} \quad Y = 0, \pi. \tag{35f}$$

These equations contain two different types of singularity. One of them we have already seen in §9. When one attempts to obtain a solution in the usual fashion, the Fourier transform of ψ_2^* turns out to be singular at $k = 0$, with the degree of singularity being $O(k^{-2})$. This singularity is due to the contributions from the term $2\,\partial^2\psi_1/\partial x\,\partial X$ in (35*a*), and the boundary condition (35*e*). Clearly this cannot be allowed, and its elimination determines the value of $\partial\psi_1/\partial X|_{X=0}$, just as $\partial\psi_0/\partial X|_{X=0}$ was determined in §9. This boundary condition would be necessary were we intending to proceed in calculating higher-order terms.

More important, there is another singularity in the Fourier transform at $k = 0$ of order $\delta(k)$, where $\delta(k)$ is the delta function, which is caused by the terms independent of x. These terms must cancel one another or a solution is impossible, and this restriction leads to the resonance equation. After substitution for the known terms on the right-hand side of (35*a*–*c*), they reduce to

$$\nabla^2\psi_2^* = e^z \cos Y[(-C''-2\lambda C)\cos t+(-D''-2\lambda D)\sin t], \tag{36a}$$

$$\frac{\partial\psi_2^*}{\partial z}-\frac{\partial\eta_2^*}{\partial t} = \left[\tfrac{1}{32}(C^2+D^2)+\tfrac{1}{16}(7-\surd 17)\ A^2-\frac{\partial^2}{\partial\tau\,\partial t}\right] \times \cos Y(C\cos t+D\sin t) \quad \text{on} \quad z=0, \tag{36b}$$

$$\frac{\partial\psi_2^*}{\partial t}+\eta_2^* = \left[\tfrac{7}{32}(C^2+D^2)+\tfrac{1}{16}(69-19\surd 17)A^2-\frac{\partial^2}{\partial\tau\,\partial t}\right] \times \cos Y(C\sin t-D\cos t) \quad \text{on} \quad z=0. \tag{36c}$$

These are easy to analyse by Fourier transform. ψ_2^* has the form

$$\psi_2^* = \tfrac{1}{2}z e^z \cos Y[(-C''-2\lambda C)\cos t+(-D''-2\lambda D)\sin t] + e^z\cos Y(B_1(X)\cos t+B_2(X)\sin t)+\text{forced terms}, \tag{37}$$

and substitution of this in (36*b*, *c*) finally furnishes two equations, one the $\cos t$ component and the other the $\sin t$ component. They are

$$-2\frac{\partial C}{\partial\tau}-\frac{1}{2}\frac{\partial^2 D}{\partial X^2}+JD = \tfrac{1}{4}(C^2+D^2)D, \tag{38a}$$

$$2\frac{\partial D}{\partial\tau}-\frac{1}{2}\frac{\partial^2 C}{\partial X^2}+JC = \tfrac{1}{4}(C^2+D^2)C, \tag{38b}$$

where

$$J = -\lambda+\tfrac{1}{4}(5\surd 17-19)A^2, \tag{39}$$

and the boundary conditions are given by (34).

Equations (38) can be regarded in another light as the complete Fourier transform of the original equations. By a complete Fourier transform we mean that the definition of the Fourier transform as an infinite integral is applied for the complete variable x, so that the domain of integration is $0 \leqslant x\,(\text{or } X) \leqslant \infty$, and X is not regarded as constant in the process of forming the integral. This is a non-uniform operation on the perturbation expansion since functions of X only will have transforms larger by ϵ than functions of x only. Thus, after substitution of the perturbation expansion (12) into the equations of motion, we must not immediately equate the different orders of magnitude, but must first apply the complete Fourier transform, and then equate the different orders of magnitude. From the knowledge that we have gained in earlier sections about the form of the perturbation expansion, it is easy to see that the leading-order terms in the equation that one obtains by this for the resonance component, are the same as one obtains by a Fourier (X) transform of (38). This approach is easier to understand but more difficult to formalize. It was used to obtain the scalings originally in §3.

Equations (38) are wave equations with the terms which are second derivatives with respect to X representing radiation of energy along the channel, and with the terms which follow representing phase changes in the wave caused by nonlinear interactions and by the deviation of the channel width from the perfect resonant wavelength. The boundary conditions (34) represent an energy input in the region of the wavemaker. (Remember that x is asymptotically small compared with X, so that $X = 0$ can represent a finite region on the x-lengthscale.) In fact the detailed calculations have shown that work is done on the cross-wave both by the Reynold stresses and also directly by the wavemaker itself. The former produced the H-term of (23) and the latter the K-terms of (33). The connection of the energy input to the radiation term can be seen by multiplying (38*a*) by C, (38*b*) by D, subtracting the two results, and finally integrating with respect to X from zero to an arbitrary point to get

$$\frac{d}{dt}\int_0^X (C^2+D^2)\,dX = \frac{1}{2}\left[L(C_0^2-D_0^2)-\left(C\frac{\partial D}{\partial X}-D\frac{\partial C}{\partial X}\right)_X\right].$$

The left-hand side represents the change in energy in the cross-wave in the region $[0, X]$, and the right-hand side shows that energy is fed into or extracted from the wave at $X = 0$, according to whether $L(C_0^2-D_0^2)$ is positive or negative, while energy radiates out of or into the region according to the sign of $C\partial D/\partial X - D\partial C/\partial X$ at X. In particular, in the early development of any wave when one would expect C and D to be zero at large distances the growth or decay of the wave depends entirely on the former of these two terms. Notice that there is no distributed source in this equation, which shows that, as other authors have commented, the cross-wave extracts energy only in the complex flow region near the wavemaker, and is incapable of extracting energy from the primary wave generated by the wavemaker even though this exists over much greater distances.

11. Discussion

It is interesting to compare our final equation with the work of previous authors. Since their work was done on the basis of small cross-wave amplitude we may begin by neglecting the nonlinear terms on the right-hand sides of (38). The linear equations that result are then equivalent to the integrodifferential equations studied by Mahony (1972). To show this, one takes the half-range Fourier cosine transform of the equations with respect to the variable X. The boundary conditions (34) then introduce an extra term into each equation which can then be expressed in terms of the transform variables $\bar{C}$ and $\bar{D}$ using the Fourier inversion theorem. For instance (38*a*) becomes

$$-2\frac{\partial \bar{C}}{\partial \tau}+\tfrac{1}{2}k^2\bar{D}+J\bar{D} = \frac{1}{2}\frac{\partial D_0}{\partial X} = \tfrac{1}{2}LC_0 = \frac{L}{\pi}\int_0^\infty \bar{C}\,dk.$$

These equations are the same as those studied by Mahony except for the difference in the values of the constants J and L. Using Laplace transforms, he proved in certain situations the existence of a growing mode of solution, hence showing that the solution $C = 0, D = 0$ is unstable. One can now do slightly better than this by solving the linearized differential equation, whereupon one can deduce that Mahony's unstable mode is a solution of separable form, namely

$$C = B\{[1-i)\,q^{\frac{1}{2}}e^{-pX}+(1+i)\,p^{\frac{1}{2}}e^{-qX}\}\,e^{\nu t},$$
$$D = B\{(-(1+i)\,q^{\frac{1}{2}}e^{-pX}-(1-i)\,p^{\frac{1}{2}}e^{-qX}\}\,e^{\nu t},$$

where B is an arbitrary real constant, and

$$\nu^2 = \tfrac{1}{16}(L^4-4J), \quad p = (2J-4i\nu)^{\frac{1}{2}}, \quad q = (2J+4i\nu)^{\frac{1}{2}}, \tag{40}$$

with p and q having positive real parts.

Barnard & Pritchard (1972) in their experiments on cross-waves found good agreement with (40) for the growth rate of small cross-waves, even though the values of L and J that they used were derived from Mahony's work and were not actually correct. We shall show that this agreement is fortuitous in that, for the experiment performed, the terms not discovered by Mahony happen to have numerically small values and so can be neglected without serious error.

Barnard & Pritchard's wavemaker was a plane which pivoted about a line on the bed of their channel with maximum angular displacement θ. For convenience we will only consider their second mode experiment where the channel depth $d = 16.4$ cm and the frequency of oscillation was varied close to the value $2\sigma = 28.3\ \mathrm{s}^{-1}$. Then the lengthscale $g/\sigma^2 = 4.90$ cm is reasonably less than the channel depth and we can expect our solution for infinite depth to be applicable. The maximum wave amplitude $a = d\theta$, so that $\epsilon = a\sigma^2/g = 3.35\theta$ cm, while the wavemaker itself is described by the equation

$$f(s) = 1 + \frac{g}{d\sigma^2}s, \quad -\frac{d\sigma^2}{g} \leqslant s \leqslant 0.$$

If we now express equation (40) for the growth rate in dimensional form we have

$$\nu\epsilon^2\sigma = \sigma\left\{\epsilon^4(H+K)^4 + \left[\epsilon^2 M - \left(\frac{\omega^2}{\sigma^2} - 1\right)\right]^2\right\}^{\frac{1}{2}}, \tag{41}$$

where $\omega^2 = n\pi g/b$ must be within $O(\epsilon^2)$ of σ^2, and in practice can only be found to that accuracy by using the empirical results, H is defined by (23.2) and can be evaluated as $H = (2d\sigma^2/g) - 2 = 6.5$, K is defined by (33*b*) and can be evaluated as

$$K \approx g/2d\sigma^2 = 0.15,$$

and

$$M = 4(5\sqrt{17} - 19)\left(\int_{-\infty}^{0} f(s)\,\mathrm{e}^{4s}\,\mathrm{d}s\right)^2$$

$$\approx 6.46\left[1 + \frac{g}{16d\sigma^2}\right]^2 = 0.47.$$

Now the terms that were not found by Mahony are those represented by K and M. It can be seen, however, that these only make a relatively small contribution to (41) and that it is acceptable to neglect them, leaving

$$\nu\epsilon^2\sigma = \omega\left\{\left(\frac{\sigma^2 d}{g}\right)^4\theta^4 + \left(\frac{\omega^2}{\sigma^2} - 1\right)^2\right\}^{\frac{1}{2}}, \tag{42}$$

which was the expression to which Barnard & Pritchard successfully fitted their experimental data. Thus we may take their quite excellent agreement as confirming our own theoretical expressions. (To be precise, in order to fit their data they had to add on to the growth rate $\nu\epsilon^2\sigma$ an arbitrary constant which they determined experimentally. This constant had a negative sign and so represented a damping effect on the wave, presumably due to effects not considered in the analysis.)

Finally it should be mentioned that equations (38) can also be shown to agree with those of Garrett, although this is a lengthy procedure. Garrett assumed that the tank was short with the cross-wave not varying a great deal along its length. Thus one must first average a linearized version of (38) along the whole tank, using (34) to evaluate the integral of the second derivatives with respect to X. Now Garrett's final (Mathieu's) equation is non-uniform, containing terms $O(1)$ and also $O(\epsilon)$. Hence to compare this equation with ours one must first apply uniform expansion techniques to his equation. Finally, converting notations between the papers, one obtains the same equation, except for the same differences due to neglected terms that we have already noted in Mahony's paper.

At the request of one of the referees, I made a numerical study of (38) with the nonlinear terms present. To do this values had to be assumed for the unknown constants. Since J, which depends so critically on channel width, would be almost impossible to determine in a real experiment, I took the easiest option and set $J = 0$ throughout. With J zero it is possible to set $L = 1$ without loss of generality since this value can always be obtained by suitable scalings. It seemed simpler to avoid the problems that would be associated with an infinite channel, and to work with a channel of finite length (finite on the X-lengthscale, although long on the x-lengthscale). The end of the channel was chosen arbitrarily to lie at $X = 1$. As boundary conditions at that point I selected

$$\frac{\partial C}{\partial X} = 0, \quad \frac{\partial D}{\partial X} = 0 \quad \text{at} \quad X = 1,$$

which at the time I thought would represent a fixed vertical wall, acting as a wavemaker of zero amplitude so that 'L' $= 0$. This interpretation may not be correct, however, since a wall would reflect the primary wave, and such a reflection has not been taken into account in my analysis. Nevertheless these conditions are among the simplest that one can consider.

As initial conditions I took

$$C = D = 0.03\lambda(2+2X-X^2) \quad \text{at} \quad \tau = 0.$$

The purpose of the X-dependence was to allow the initial conditions to satisfy the boundary conditions at both $X = 0$ and $X = 1$. The parameter λ represents the magnitude of the initial value. The linear and nonlinear terms in the equations become comparable when C and D take values close to unity. This occurs at the initial moment for a value of λ round about 10. Thus we may refer to the starting conditions as small if $\lambda \ll 10$ and large if $\lambda \gg 10$.

The equations (38) were solved by a finite-difference approximation. Since the explicit approximation scheme was unstable for all step lengths a semi-implicit (Crank–Nicolson) scheme was employed. However, this would have been expensive if applied directly to the nonlinear terms, since it would have involved alteration of the inversion matrix at each step, thus requiring repeated diagonalization in order to achieve solutions. Instead the nonlinear terms were treated as explicit terms, so that the matrix one had to invert depended only on the linear terms, and so was always the same. Hence it could be diagonalized once and for all at the beginning of the program. Then to obtain the semi-implicit approximation for the nonlinear terms a corrector step was added, allowing the desired form to be approached iteratively. For instance, $(C^2+D^2)C$ was replaced by

$$\tfrac{1}{2}[(C_{i,j}^2+D_{i,j}^2)\,C_{i,j}+({}_\alpha C_i^2+{}_\alpha D_i^2)\,{}_\alpha C_i],$$

where α is a counter for the iterations. Initially one starts with an explicit approximation, i.e. ${}_0C_i = C_{i,j}$, ${}_0D_i = D_{i,j}$, and the equations are then solved to produce a first interim solution for $C_{i,j+1}$ and $D_{i,j+1}$, say C_i' and D_i', and the equations are resolved with this new explicit term. This procedure is then repeated until the maximum change over the whole interval is less than some specified error value (10^{-5} was selected for this purpose). Provided that the iterations converge, the final solution approaches a limiting value which by definition is $C_{i,j+1}, D_{i,j+1}$. Substitution of this limiting value for ${}_\alpha C_i$, ${}_\alpha D_i$ in the representation of the nonlinear terms shows that, in the limit, these terms have the desired semi-implicit form.

It is fairly easy to show that the numerical scheme was consistent, the approximation scheme being accurate to second order. It was also desirable to perform a stability analysis, but this was more difficult because of the presence of the nonlinear terms. However, if one arbitrarily neglects these terms, so as to obtain linear equations, then von Neumann's method can be used. Essentially this is a local Fourier analysis of the numerical method, determining whether harmonic waves present in the solution

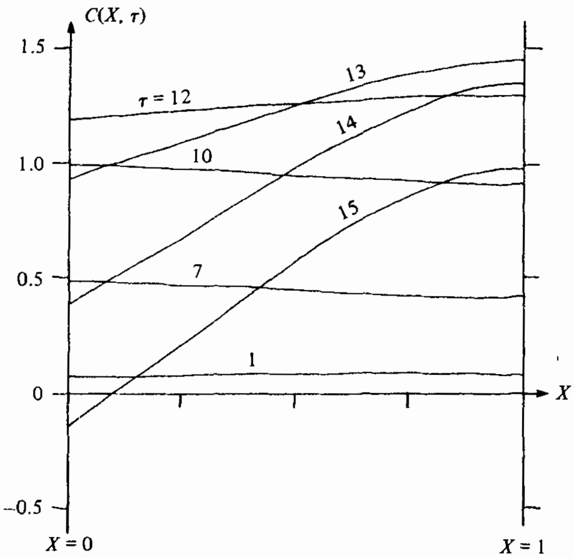


FIGURE 1. Wave profiles at different times for $\lambda = 1$.

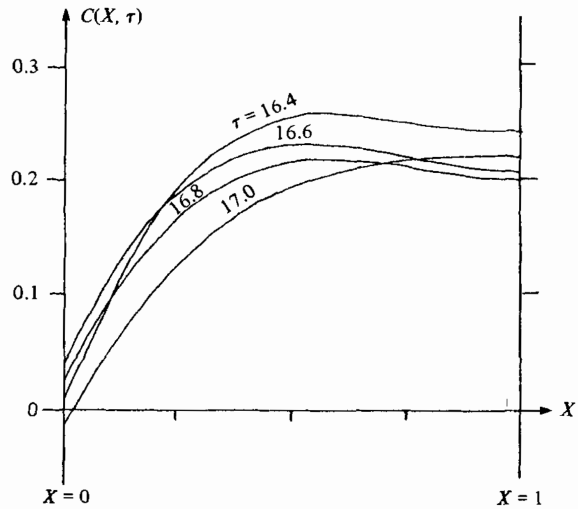


FIGURE 2. Wave profiles at different times for $\lambda = 1$.

become amplified with time or whether they decay. It was found that the semi-implicit scheme was neutrally stable to the growth waves of any wavelength. Further, one could also consider schemes where the averaging of the second derivatives in x were not 1 : 1. When the averagings were biased towards a more explicit approximation scheme, the waves were found to grow with time, and, when they were biased towards being fully implicit, they were found to decay with time. Now the former behaviour cannot be permitted because this means that the solution is unstable. However, since one would expect the equations to have solutions of wavelike form, the second type of behaviour, which would damp out waves, is also undesirable. Hence the precisely 1 : 1 semi-implicit formulation actually used would seem to have the most nearly ideal behaviour in this respect.

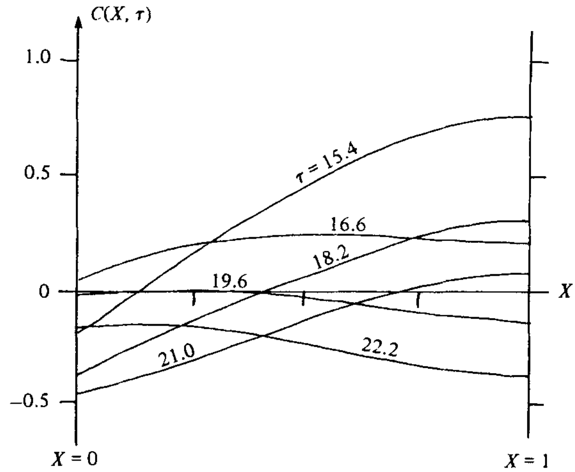


FIGURE 3. Wave profiles at different times for $\lambda = 1$.

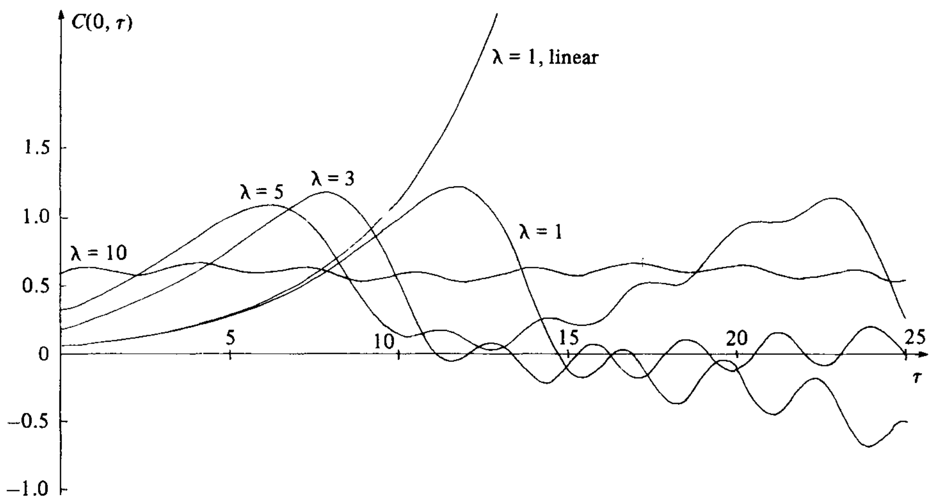


FIGURE 4. Graphs of C at $X = 0$ against τ for different initial values.

A similar analysis can be performed with the nonlinear terms present by linearizing the equations about a solution assumed known and by looking for the growth of a small error term. However, for the von Neumann method to apply one must treat the known solution locally as if it were a constant. Results so obtained are thus only applicable to waves which are short compared with the lengthscale on which the solution is varying. Within this limitation, one again finds that the semi-implicit approximation is neutral to the growth or decay of waves.

We now discuss the numerical results. Figures 1–3 show the x-dependence of the coefficient C at various times for the starting value $\lambda = 1$. It can be seen that once it develops away from its nearly uniform initial state it develops into a rather tame, oscillating wave. The variable D has a similar behaviour. Also other initial conditions for C and D lead to similar x-profiles. Since our intention is only to show the form of the solution we have not reproduced these other curves.

Figure 4 shows the variation with time of the value of C at $x = 0$ for various starting values of λ. When λ is small, the nonlinear terms in the equation are also small, and

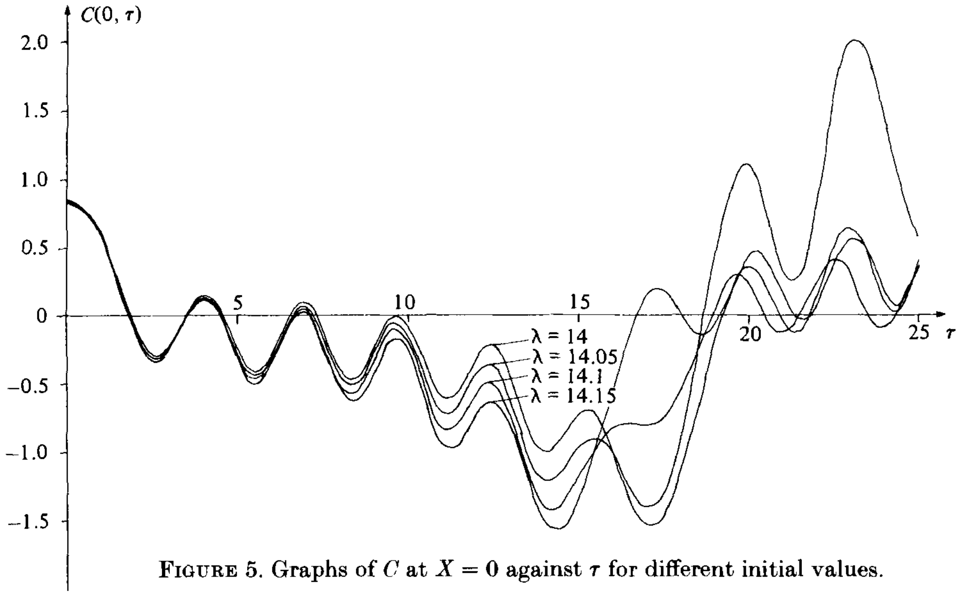


FIGURE 5. Graphs of C at $X = 0$ against τ for different initial values.

the initial behaviour should be governed by the linearized equations. This was confirmed for the case $\lambda = 1$ by solving the linearized equations numerically with the same starting conditions. There was a check on this latter curve since at large times it must increase exponentially at a rate that can be found by theoretical calculation. The exponent can also be estimated from the numerical curve and agreement was excellent. This curve is also plotted in figure 4. It can be seen that the nonlinear curve follows it closely until the value of $C(0, t)$ exceeds 0.5, when the nonlinear terms reach parity with the linear terms and modify the solution. The subsequent behaviour then appears chaotic, using this term in its technical sense of an apparently random but bounded oscillatory type of behaviour. A typical period of these oscillations is about three time units.

A similar type of behaviour can be seen in figure 4 for the starting values $\lambda = 3$ and $\lambda = 5$. There is an initial smooth exponential growth followed by chaotic behaviour. The case $\lambda = 10$ provides a watershed with the nonlinear terms neither negligible nor dominant at the initial moment. Some solution curves for larger λ when the nonlinear terms are immediately dominant are given in figure 5. Chaotic behaviour is immediately obtained. Such curves were difficult to obtain, however, since the larger the nonlinear terms the worse was the convergence behaviour of the iteration scheme described above, and hence the finer the x- and t-steplengths needed to bring this under control.

The relatively smooth behaviour of the $\lambda = 10$ curve is interesting since it performs such regular small-amplitude oscillations. Larger starting values of λ lead to large oscillations. Smaller starting values seem to produce solutions which 'overshoot' the $\lambda = 10$ solution and, having once obtained large values, also undergo large oscillations. Thus one is led to speculate that some sort of stable behaviour might be associated with intermediate initial values.

Figure 5 also demonstrates another typical property of chaotic behaviour, that small changes in the initial conditions do not produce changes in the solution that are uniformly small with time. The behaviour illustrated is typical. The curves diverge very slowly from one another until some critical difference is reached, whereupon a rapid divergence follows, and the subsequent paths are entirely uncorrelated. The smaller the change in the initial conditions, the larger is the time before significant divergence occurs. Conversely the larger the value of λ the more sensitive is the solution to changes in λ, i.e. the solution becomes non-uniform more rapidly for large starting values.

This non-uniformity had an unfortunate consequence on the numerical method. Small changes in the mesh sizes of x and t produced small changes in the solution, but these small changes would slowly grow with time, reach a critical amount, and then switch onto another curve entirely. This non-uniformity in behaviour was much more sensitive to changes in Δt than to changes in Δx. To illustrate, most of the runs of figure 5 were done with $\Delta x = 0.0125$ and $\Delta t = 0.01$ on the time range $[0, 50]$. A run would then be validated by repeating the numerical integration but with $\Delta t = 0.005$. It would typically be found that the curves remain close until $t = 30$, whereupon rapid divergence of the same type as figure 5 would ensue. Thus the first run could only be considered as correct in the range $[0, 30]$. If a further validation was performed, with $\Delta t = 0.00125$ say, then the second and third curves might remain close until $t = 35$, say, followed again by chaotic divergence. Changes in Δx had a similar, but much weaker, effect.

Now one point of interest about the equations is whether the solution always remains bounded or whether it can diverge to infinity. I had hoped to investigate this by following the solution curves to very large time values. Non-divergence would not prove stability, of course, but divergence would definitely disprove it. Unfortunately the above difficulty imposed a severe limitation on this programme. It was often found that the solution diverged, especially on earlier runs when much coarser mesh sizes were used, but it was *always* found that these divergences were outside the solution's range of validity, and that they disappeared when finer meshes were used. Thus the stability question still remains open.

Finally some comments of a general nature. Many people have pointed out to me that (38) is an example of the nonlinear Schrödinger equation. If so desired both (38) and (34*c*) can be expressed in terms of the complex variable $C + \mathrm{i}D$ and its complex conjugate. However, it is not clear to me what useful conclusions can be drawn from this. As exemplified by the instability condition (40) the physical nature of the solution is strongly influenced by the self-interaction represented by (34*c*). Most published work on the NLS involve waves on an infinite domain with no boundary conditions at all. A thorough study of the NLS on a semi-infinite expanse was recently published by Aranha, Yue & Mei (1982), but even there the waves were driven by an explicit nonhomogeneous term in the boundary conditions, rather than by the implicit, homogeneous form of (34*c*).

While the present equations could be solved by sufficient numerical effort, there is the question of whether this is worthwhile, given that there is no likelihood of the results agreeing with experiment. For (38) and (34*c*) only represent the slow evolution of a cross-wave due to modulation by nonlinear effects. As discussed by Chu & Mei (1971), it must be borne in mind that, because the changes are slow, other forces which one would normally consider negligible, such as viscosity and surface tension, also have a long time in which they can act upon and affect the wave profile. That such forces indeed cannot be neglected even in the linearized case is shown by the empirical damping coefficient that had to be introduced to get agreement with the experiments of Barnard & Pritchard (1972). Thus the present work should be considered as a preparatory study. It deals with the most complicated of the modifying forces, but the effects of the other forces must be synthesized with it if a full description of cross-waves is to be obtained.

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